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“Games of No Chance”: Combinatorial Game Theory

1.1 **Introduction**

In the programming behind games, **Combinatorial Game Theory** (CPT) stands out for its usage of mathematics to win. Contrary to the Classical Game Theory and its probabilistic approach, CPT does not rely on chance or luck, as each player has perfect information. It is used to analyze the game and determine the best strategies to win. Developed in the 20th century, CPT has even proved in combinatorial games that “ either *one* player can force a win, or *both* players can force a draw” (Legner 2). The algorithm of it in games like Nim creates interesting results and ways to play by harnessing mathematics.

Definition 1.1 A *combinatorial game* is a game played between two opponents. The game consists of the following attributes:

(1) A set of possible positions.
(2) A move rule indicating the positions Player 1 and Player 2 can move to during their respective turns.
(3) A win rule indicating a set of terminal positions that signifies the end of the game. With each terminal outcome, there is an associated outcome: Player 1 wins and Player 2 loses (+-), Player 1 loses and Player 2 wins (-+), or a draw (00).

Definition 1.2 An *impartial game* is a game played between two opponents. Its rules are as follows:

(1) Two players move alternatively.

(2) All moves and options are specified by the rules.

(3) Finitely many positions, the game will end when one player cannot move (no draws or loops).

(4) Perfect information for both players.

Impartial games work well with the CPT, as will be seen through some classic game examples.

Definition 1.3 A *misère* (French for "destitution") game is when the player unable to move wins, which is in contrast to the "normal play condition" where the player unable to move loses.

1.2 **Nim**

With its origins from China, Nim was given its name by Professor Charles Bouton at Harvard University in 1901. "Nim" means to steal or take away. In the game, two players will take turns removing, for example, matchsticks from distinct piles. The player can remove as many matchsticks they want from the *same* pile in a turn. The loser has to take the last remaining matchstick. This is a *misère* game illustrated in **Figure 1**.

Playing the Game

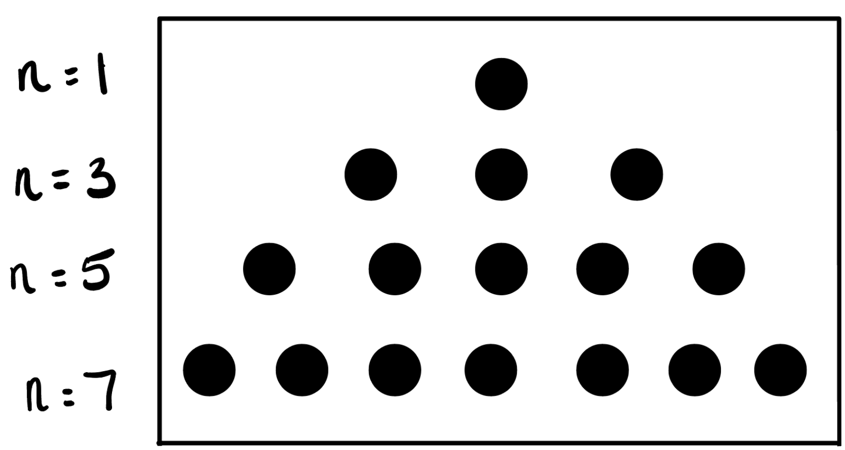

Figure 1. Nim Game set-up of 4 rows of 1, 3, 5, & 7

In order to win Nim, you must take away as many matchsticks as possible so the "Nim Sum" of the rows remain zero. This "Nim Sum" is in fact the exclusive or (**xor**) addition, symbolized by $\oplus$. The logic behind xor is that a or b can be true, but not both:

a xor b = [(a or b) and (not(a and b))]   OR $a \oplus b = [(a \vee b) \wedge (\neg(a \wedge b))]$

By counting the matchsticks in each row, they can be converted to multiples of 4, 2, and 1. Next, cancel the pairs of equal multiples. Then, add what is left. The game begins in **Figure 2**.

Row1 = **1**	= 1 x **1** = 1	=			~~1~~
Row2 = **3**	= 1 x **2** + 1 x **1**	=		~~2~~	~~1~~
Row3 = **5**	= 1 x **4** + 1 x **1**	=	~~4~~		~~1~~
Row4 = **7**	= 1 x **4** + 1 x **2** + 1 x **1**	=	~~4~~	~~2~~	~~1~~
Total of UNPAIRED multiples		=	**0**	**0**	**0**

Figure 2. Game begins with the Nim Sum at 0.

With the Nim Sum at zero, there are zero unpaired multiples of 1, 2, and 4. By playing the game and keeping a configuration of zero, it will guarantee a higher chance of winning. When the opponent moves, it leaves this configuration in **Figure 3**:

Row1 = **1**	= 1 x **1**	=			~~1~~
Row2 = **3**	= 1 x **2** + 1 x **1**	=		~~2~~	~~1~~
Row3 = **5**	= 1 x **4** + 1 x **1**	=	~~4~~		~~1~~
Row4 = **5**	= 1 x **4** + 1 x **1**	=	~~4~~		~~1~~
Total of unpaired multiples		=	**0**	**1**	**0**

Figure 3. Opponent moves, making the Nim Sum 1.

This leaves the player at a disadvantage, so the optimal move is to remove 2 matchsticks from the 2nd row as seen in **Figure 4**:

Row1 = **1**	= 1 x **1**	=			~~1~~
Row2 = **1**	= 1 x **1**	=			~~1~~
Row3 = **5**	= 1 x **4** + 1 x **1**	=	~~4~~		~~1~~
Row4 = **5**	= 1 x **4** + 1 x **1**	=	~~4~~		~~1~~
Total of unpaired multiples		=	**0**	**0**	**0**

Figure 4. Player moves, making the Nim Sum 0 again.

Using the xor operation, it can be used as a winning strategy. Looking at it in the bitwise operation, it will return a 1 in a bit position if bits of one but not both operands are 1's. **Figure 5** shows an example of converting the bits of base 2 to base 10:

101_2 -> 5_{10} number A
111_2 -> 7_{10} number B

010_2 -> 2_{10} The "Nim sum" of numbers A and B, $5 \oplus 7 = 2$

Figure 5. Nim Sum was found using bitwise operations.

By using this method, it is possible to win a game of Nim against a computer by utilizing the language of the computer: binary. As well, to move first would be a disadvantage as it increases the chance of selecting the last matchstick. I have not won against it yet, but hope to in the near future.

Conclusion

Combinatorial Game Theory is based on the mathematics of an impartial game. By looking at a Nim, it proves to be a game of no chance. Creating strategies with a thorough understanding of xor addition, or referred to as the "Nim Sum," a win can be calculated. As AI begins to develop, I am curious of the future of more combinatorial games where there new positions generated in real time. Would there still be perfect information? Or would the AI be able to compute every possible solution? A comparison between CPT and AI is comparable to mathematics and engineering: “The mathematicians' main goal is understanding; the engineer's main goal is a functioning system” (Berkeley). Both of these systems an work from and with each other. As long as the AI is taught to play, it could create games beyond a scope we currently know.

References

"Combinatorial Game Theory Background." *University of California, Berkeley,* https://math.berkeley.edu/~berlek/cgt/cgt-info.html. Accessed 28 April 2024.

Davis, Dean J., "The Game of Nim." *MAT Exam Expository Papers*. 9. https://digitalcommons.unl.edu/mathmidexppap/9. Accessed 28 April 2024.

"How to Win at the Nim Game." *Archimede's Laboratory*, https://www.archimedes-lab.org/How_to_Solve/Win_at_Nim.html#. Accessed 28 April 2024.

Legner, Phillip. "Combinatorial Game Theory: Analysis of the Game of Nim and some interesting Variants." *Mathigon*, https://mathigon.org/world/resources/Game_Theory/Combinatoial_Game_Theory.pdf. Accessed 28 April 2024.

"Nim: Ancient Number Game." *Britannica*, https://www.britannica.com/topic/nim. Accessed 28 April 2024.

Yuan, Jennifer and Shreya Shin. "Game Theory." *Massachusetts Institution of Technology*, https://math.mit.edu/research/highschool/primes/circle/documents/2021/Yuan_Sinha.pdf. Accessed 28 April 2024.